An Analysis of the DD Algorithm for Group Testing with Size-Constrained Tests

Nelvin Tan and Jonathan Scarlett

National University of Singapore (NUS)

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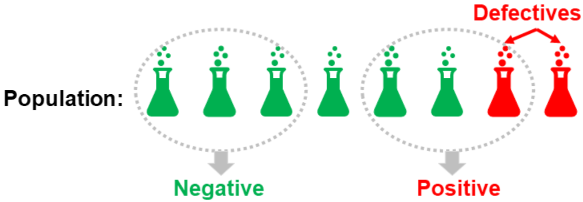
Introduction



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  - Medical testing (e.g., COVID-19)
  - Data science
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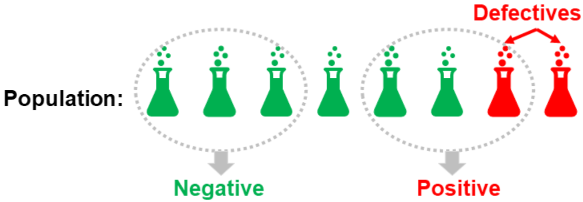

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Setup

- Population of n items labelled $\{1, \dots, n\}$.
- Defective set $\mathcal{S} \subset \{1, \dots, n\}$, where $k = |\mathcal{S}| = \Theta\big(n^{\theta}\big)$, for $\theta \in [0, 1)$.
- We consider the following settings:
  - **Combinatorial** prior: Defective set $\mathcal{S}$ chosen uniformly among all sets of size k.
  - **Non-adaptive:** Test pools are designed in advance.
  - **Noiseless:** Get a +ve test outcome if there is least one defective item, and a -ve outcome if there is no defective item.
  - **Small error probability recovery:** Produce $\widehat{\mathcal{S}}$ such that

$$\mathbb{P}[\widehat{\mathcal{S}} \neq \mathcal{S}] \to 0 \text{ as } n \to \infty,$$

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Sparsity-Constrained Group Testing

- Tests are size-constrained and thus contain no more than ρ items per test.
- **Applications:** Testing equipment may be limited in the volume of samples it receives, or large pools may be unsuitable due to dilution effects,

- Previous work in literature shows that $\rho \in \Theta\left(\frac{n}{k}\right)$ is required to attain optimal scaling laws for the unconstrained setting.
- Hence, we are interested in the regime $\rho \in o\left(\frac{n}{k}\right)$, and more specifically $\rho = \Theta\left(\left(\frac{n}{k}\right)^{\beta}\right)$, for $\beta \in [0, 1)$.

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Previous Result

Under our setup (recall $k = \Theta(n^\theta)$ and $\rho = \Theta\big(\big(\frac{n}{k}\big)^\beta\big)$), we have:

- **Converse:** $\frac{1-6\epsilon}{1-\beta} \cdot \frac{n}{\rho}$.
- **Achievability:** $\left\lceil \frac{1+\epsilon}{(1-\theta)(1-\beta)} \right\rceil \cdot \left\lceil \frac{n}{\rho} \right\rceil$.
- **Improved converse (when $\beta = 0$):** $\max\left\{\left(1 + \left\lfloor \frac{\theta}{1-\theta} \right\rfloor\right)\frac{n}{\rho}, \frac{2n}{\rho+1}\right\}$.
- **Improved achievability (when $\beta = 0$):** $\max\left\{\left(1 + \left\lfloor \frac{\theta}{1-\theta} \right\rfloor\right)\frac{n}{\rho}, \frac{2n}{\rho+1}\right\}$.

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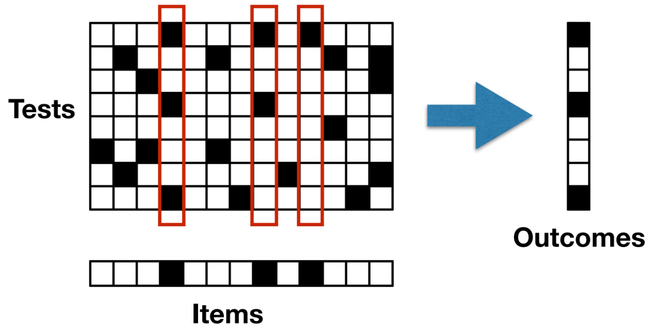
Outline

Randomized Test Design

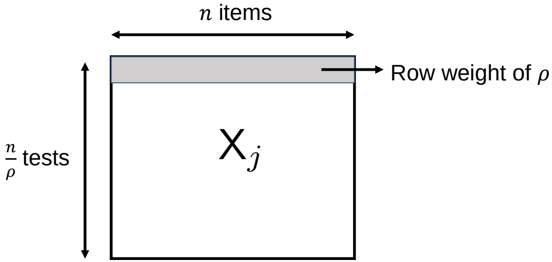
Example of a random test matrix:



- The i-th column represents the tests that the i-th item participates in.
- The i-th row determines the items in the i-th test pool.

Randomized Test Design

Uniformly sample the sub-matrix X_j:



- Sample c sub-matrices $(\mathsf{X}_1, \dots, \mathsf{X}_c)$ independently.
- Form test matrix by concatenating $\mathsf{X}_1, \dots, \mathsf{X}_c$ vertically, giving us a doubly-constrained matrix.
- Column weight restriction is not strictly imposed by the testing constraints but helps in avoiding "bad" events where some items are not being tested.

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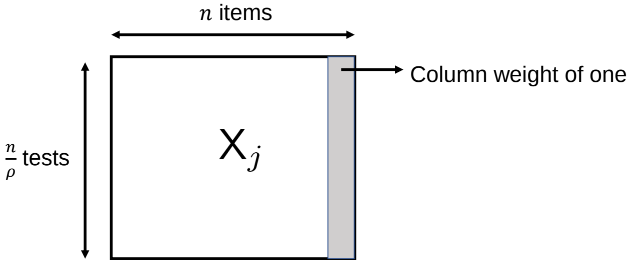

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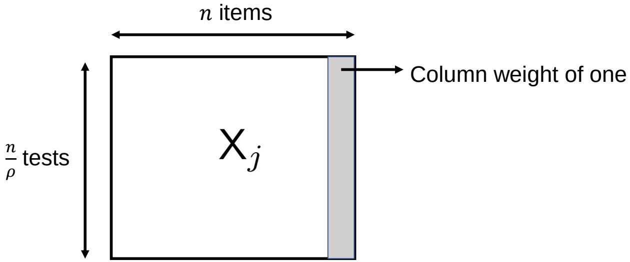

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Definite Defectives (DD) Algorithm

Some useful observations:

- Any item in a negative test is definitely non-defective (DND).
- All other items can (initally) be considered possibly defective (PD).
- If a test contains only one PD item, then that item is definitely defective (DD).

Algorithms:

- **COMP (previous):** Declare DND items to be non-def. and the rest def.
- **DD (this talk):** Declare DD items to be def. and the rest non-def.

Main Result

Theorem

For $k = \Theta\big(n^{\theta}\big)$*, with* $\theta \in [0,1)$*, and* $\rho = \Theta\Big(\big(\frac{n}{k}\big)^{\beta}\Big)$*, with* $\beta \in [0,1)$*, for any integer* c *satisfying:*

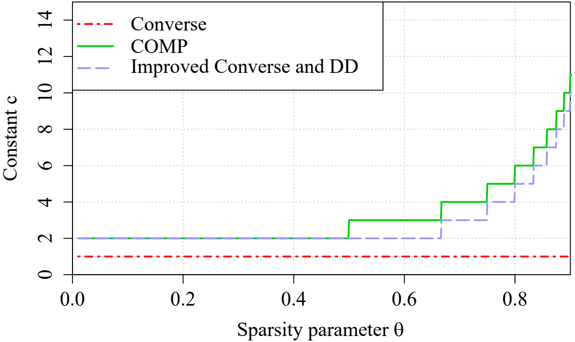
- *If* $\theta \geq \frac{1}{2}$: $c > \frac{\theta}{(1-\theta)(1-\beta)}$ *and* $c \geq \frac{2-\beta}{1-\beta}$*;*
- *If* $\theta < \frac{1}{2}$*:* $c \geq \frac{1-\theta\beta}{(1-\theta)(1-\beta)}$*;*

the DD algorithm with $\frac{cn}{\rho}$ *tests recovers the defective set* $\mathcal{S}$ *with asymptotically vanishing error probability.*

- This is an improvement over previous acheivability results.

Main Result

Plot for $\beta = 0$:



- For $\beta = 0$, our result coincides with the improved achievability result.
- For each β shown (0, 0.25, and 0.75), there are strict improvements over COMP (previous work), and the gap widens for $\beta = 0.75$.

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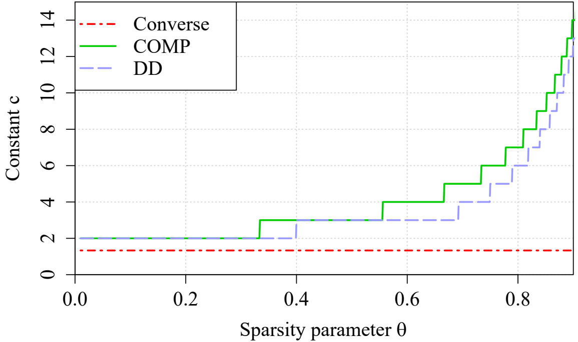
Plot for $\beta = 0.25$:



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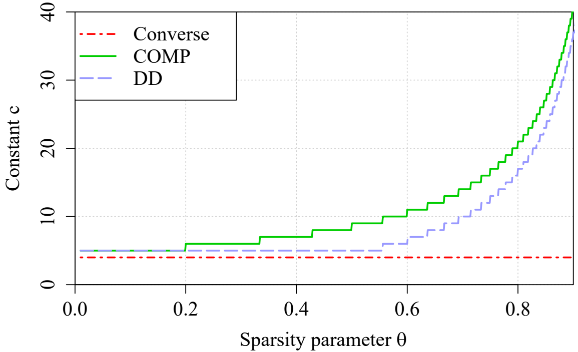

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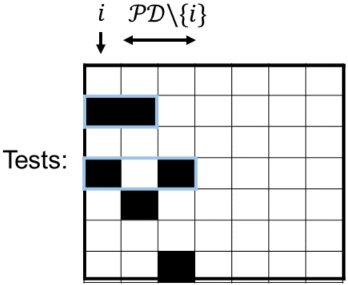
Analysis Outline of the DD Algorithm

- **DD algorithm:**
  - Declare all DD items as defective and the rest non-defective.
  - DD algorithm makes a mistake if any defective item i is masked by $\mathcal{PD} \setminus \{i\}$ in the test matrix.



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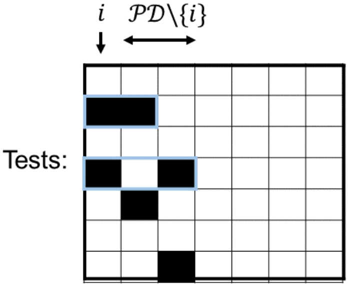

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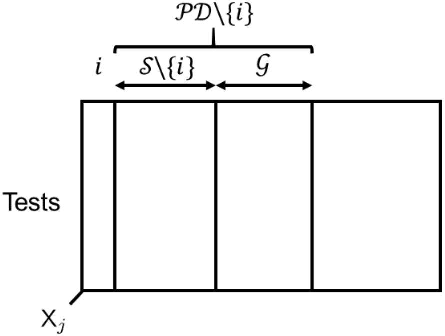
- For a given X_j and a defective item i, the error event can be simplified in the following manner:



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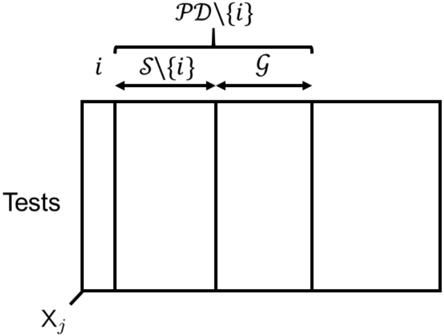

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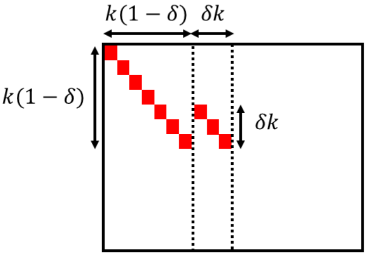
Analysis Outline of the DD Algorithm

- Conditioned on the #positive tests $k(1-\delta)$ and the size of $\mathcal{G}$, we can upper bound $\mathbb{P}\big[i \text{ masked by } \mathcal{S}\setminus\{i\}\big] + \mathbb{P}\big[i \text{ masked by } \mathcal{G} | i \text{ not masked by } \mathcal{S}\setminus\{i\}\big]$.
- This gives us the upper bound $\frac{2\delta k}{k} + \frac{|\mathcal{G}|}{k(1-2\delta)}$.



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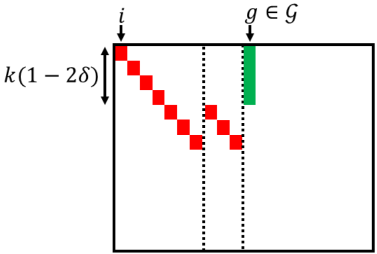

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- We have an upper bound on the error probability for a given sub-matrix X_j and defective item i.
- Recall that we need the error probability to hold for the entire test matrix and for any defective item.
  - **Extend to entire test matrix:** Repeatedly multiply c times (by independence of sub-matrices).
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Summary

- We used
  - independent doubly-constrained sub-matrices;
  - the DD algorithm.
- We showed that this improves the constant term in the achievability result.