Sparse Group Testing:

Bounds and Algorithms

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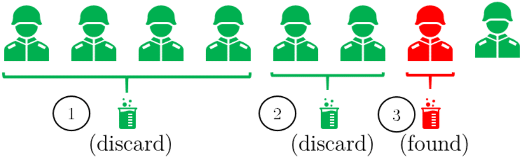
Motivation

- Begin in 1943, where soldiers were tested for syphilis by drawing blood
- Robert Dorfman's key insight: reduce number of tests by pooling
- **Central problem:**
  - How many tests are required to accurately discover the infected soldiers?
  - How can it be achieved?



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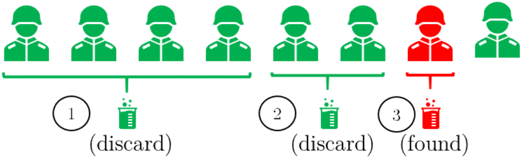

Applications

- **Medical testing:** COVID-19, by pooling Ribonucleic acid (RNA) samples [Yelin et al., 2020]

- **Some other applications:**
  - Biology
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  - Data science

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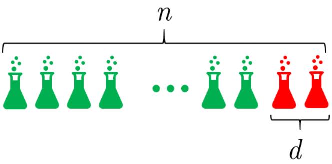
Group Testing Setup



- In this talk:
  - n items labelled $\{1, \ldots, n\}$ that produces binary outcomes when tested
  - Defective set $\mathcal{D} \subset \{1, \ldots, n\}$, where $d = |\mathcal{D}| \in o(n)$
  - Combinatorial prior: Defective set $\mathcal{D} \sim \text{Uniform}\binom{n}{d}$ (i.e, d out of n items with uniform prior)
  - Noiseless testing: negative outcome $\Rightarrow$ all items in pool are non-defective; positive outcome $\Rightarrow$ at least one item in pool is defective
  - Distinction between adaptive and non-adaptive testing

Recovery Criteria

- Error probability (exact recovery)

$$P_e = \mathbb{P}(\widehat{\mathcal{D}} \neq \mathcal{D})$$

- We study conditions on number of tests T for $P_e \to 0$ as $n \to \infty$
  - Information theoretic lower bound
    - Minimum number of tests T for $P_e \to 0$
  - Upper bound from algorithm
    - Maximum number of tests T our algorithm needs for $P_e \to 0$

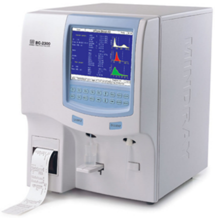
Sparse Group Testing

Testing procedure is subjected to one of the following:

- Items are finitely divisible and thus may participate in at most γ tests
- Tests are size-constrained and thus contain no more than ρ items per test

Example:

- Divisibility constraint: finite amount of blood per soldier
- Size constraint: limitations on volume capacity of the machine



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- We are interested in:
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Previous Work on Non-Adaptive Setting

For some error probability $P_e = \mathbb{P}(\widehat{\mathcal{D}} \neq \mathcal{D}) \leq \epsilon$:

Constraint type	Scaling regime	Tests required
γ-divisible items	$d \in \Theta(n^\theta), \theta < 1$ $\gamma \in o(\log n)$	$T > \gamma d \left(\frac{n}{d}\right)^{(1-5\epsilon)/\gamma}$ $T < \left\lceil \gamma d \left(\frac{n}{\epsilon}\right)^{1/\gamma} \right\rceil$
ρ-sized tests	$d \in \Theta(n^\theta), \theta < 1$ $\rho \in \Theta\big((n/d)^\beta\big), \beta < 1$ $\epsilon = n^{-\zeta}, \zeta > 0$	$T > \left(\frac{1-6\epsilon}{1-\beta}\right)\frac{n}{\rho}$ $T < \left\lceil \frac{1+\zeta}{(1-\alpha)(1-\beta)} \right\rceil \left\lceil \frac{n}{\rho} \right\rceil$

Table: Previous results (non-adaptive setting)

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Outline

1. Adaptive Setting for γ-Divisible Items

2. Non-Adaptive Setting for γ-Divisible Items

3. Conclusion

1. Adaptive Setting for γ-Divisible Items

Lower Bound Result

Adaptive setting: test pools are designed sequentially, and each one can depend on previous test outcomes.

Theorem: If $d \in o(n)$, $\gamma \in o(\log n)$, any non-adaptive or adaptive group testing algorithm that tests each item at most γ times and has a probability of error of at most ϵ requires at least $e^{-(1+o(1))}\gamma d\left(\frac{n}{d}\right)^{1/\gamma}$ tests.

Improvements: We have strengthened previous lower bound by

- Improving dependence on ϵ, and
- Extending its validity to the adaptive setting.

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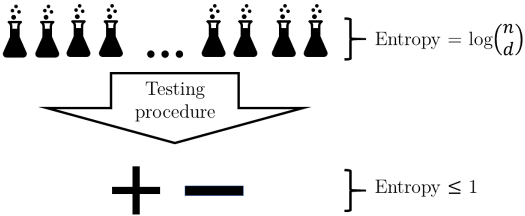
Lower Bound Result Interpretation

Theorem: We require at least $e^{-(1+o(1))}\gamma d\left(\frac{n}{d}\right)^{1/\gamma}$ tests.

Interpretation:

$\mathcal{D} \sim \text{Uniform}\binom{n}{d}$



- If every test reveals 1 bit of entropy, we need $\log\binom{n}{d} \approx d \log\left(\frac{n}{d}\right)$ tests
- Since we require much more tests, our constraint results in tests to be less informative

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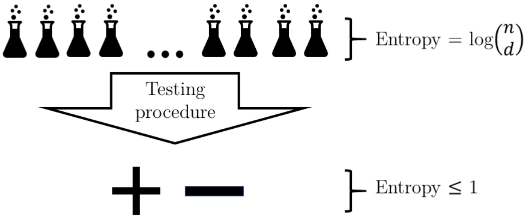

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Upper Bound Result and Algorithm

Claim: If $d \in o(n)$, $\gamma \in o(\log n)$, then there exists an adaptive group testing algorithm that tests each item at most γ times achieving $P_e = 0$ using at most $T = \gamma d(\frac{n}{d})^{1/\gamma}$ tests.

Improvements:

- Improved previous bound: $\left(\frac{n}{\epsilon}\right)^{1/\gamma} \Rightarrow \left(\frac{n}{d}\right)^{1/\gamma}$
- Matches the lower bound $T \geq e^{-(1+o(1))}\gamma d\left(\frac{n}{d}\right)^{1/\gamma}$ up to a constant factor of $e^{-(1+o(1))}$

Key idea: Can we partition the items into equal groups of ideal sizes?

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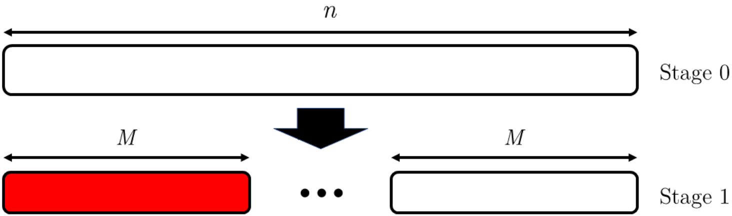
Adaptive Algorithm



- **Group sizes:** $M \to M^{1-\frac{1}{\gamma-1}} \to M^{1-\frac{2}{\gamma-1}} \to \cdots \to 1$
  - n/M splits from stage 0 to stage 1
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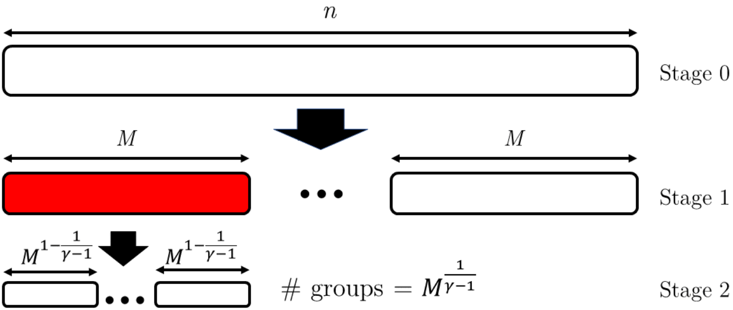

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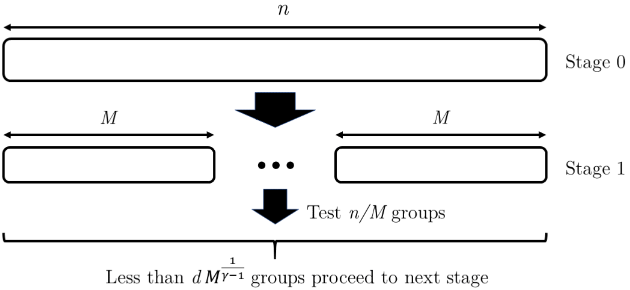
Adaptive Algorithm Analysis



- From stage 1 to stage 2: we made n/M tests

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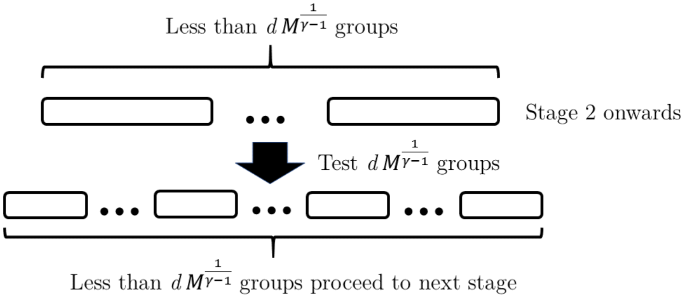
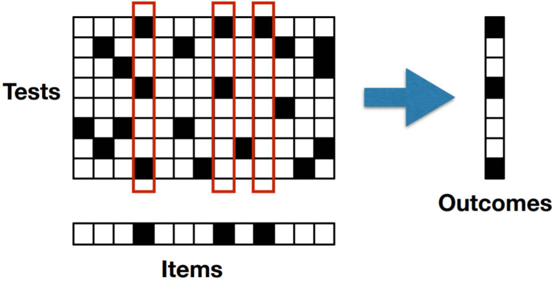


- From stage 2 onward, between any two stages (total of $\gamma - 1$): we made $dM^{\frac{1}{\gamma-1}}$ tests
- This gives us $T \leq \frac{n}{M} + (\gamma - 1)dM^{\frac{1}{\gamma-1}}$.
- Optimizing the upper bound w.r.t. M and substituting back into the upper bound, we get our result

2. Non-Adaptive Setting for γ-Divisible Items

Non-adaptive Setting

- Tests can be designed in advance
- Goal: given test matrix X and outcomes $\mathbf{y}$, estimate $\widehat{\mathcal{D}}$



- Narrower regimes:
  - $d \in \Theta(n^{\theta})$ for some sparsity parameter $\theta \in (0, 1)$
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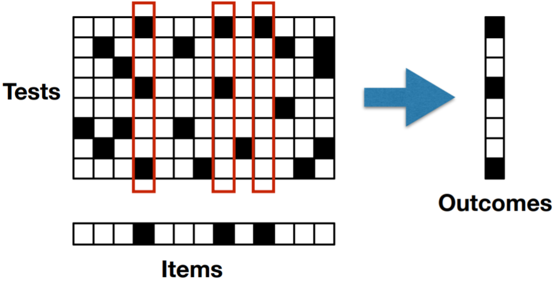

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Theorem: If $d \in \Theta(n^\theta)$ and $\gamma \in \Theta\big((\log n)^c\big)$, then any non-adaptive algorithm that tests each item at most γ times with tests $T \leq \gamma d^{1/\gamma}(d-1)(1+o(1))$, has P_e bounded away from zero.

Interpretation:

- Larger number of tests are required for some θ
- We compare our result $T \in \Omega\big(\gamma d \cdot d^{1/\gamma}\big)$ with previous result $T \in \Omega\Big(\gamma d \big(\frac{n}{d}\big)^{1/\gamma}\Big)$
- To make the difference visible, we use the variable η, where $T = \gamma d\Big(\big(\frac{n}{d}\big)^{\frac{1}{\gamma}}\Big)^{\frac{1}{\eta}}$
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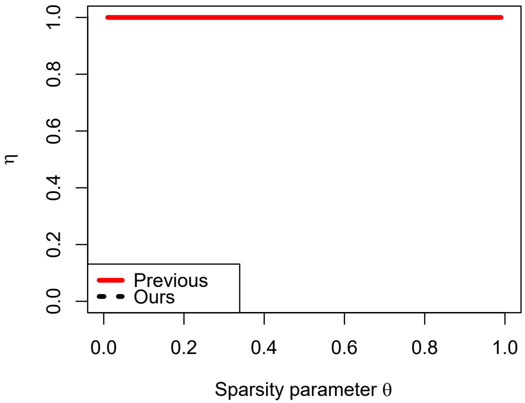

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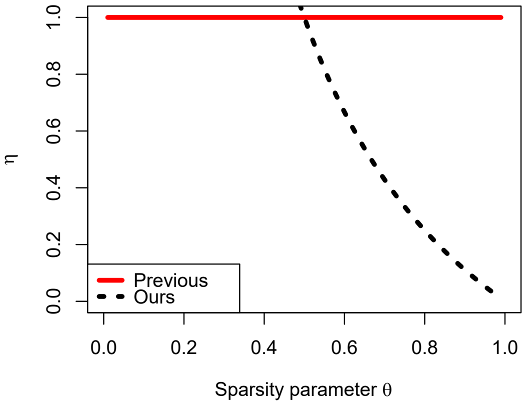

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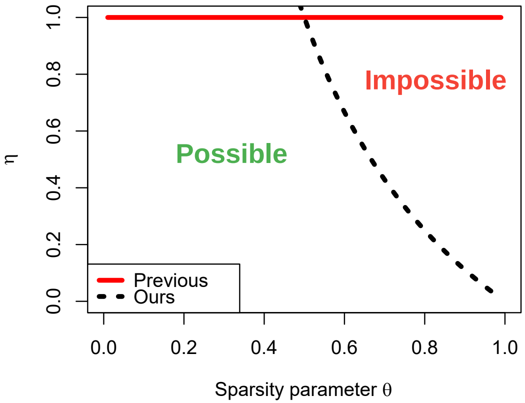

Definite Defectives (DD) Algorithm

Some useful observations:

- Any item in a negative test is definitely non-defective ($\mathcal{DND}$)
- All other items are considered (initially) as possibly defective ($\mathcal{PD}$)
- If a test contains only one item from $\mathcal{PD}$, then it is definitely defective ($\mathcal{DD}$)

Algorithm: Declare the set $\mathcal{DD}$ to be the positive and the rest negative

Claim: If $d \in \Theta(n^\theta)$, $\gamma \in \Theta\big((\log n)^c\big)$, constant $\alpha_2 \in (0,1)$, and a decaying function β_n, then we need at most

$$T = \gamma d \max\left\{ 2^{\frac{1}{\alpha_2} H_2(\max\{\alpha_2, \frac{1}{2}\})} \left(\frac{d}{\beta_n}\right)^{\frac{1}{\alpha_2 \gamma}}, 2^{1/\gamma} \left(\frac{n-d}{d}\right)^{\frac{1}{\gamma}} \left(\frac{d}{\beta_n}\right)^{\frac{1}{(1-\alpha_2)\gamma^2}} \right\},$$

to recover $\mathcal{D}$ with vanishing error probability.

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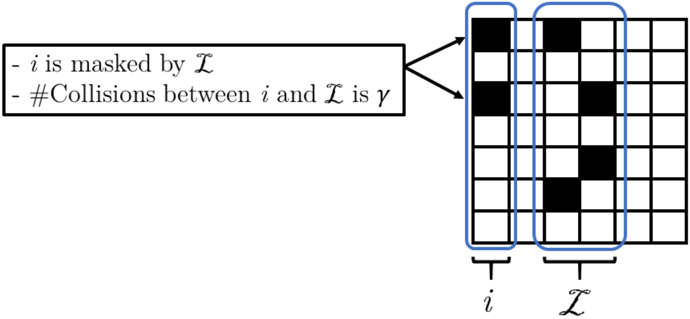
Definitions

Two Definitions:

- We say that item i is masked by set $\mathcal{L}$ if every test that includes i, also includes ≥ 1 item(s) from $\mathcal{L}$
- Number of collisions between item i and set $\mathcal{L}$ refers to the #tests that include i, and also include ≥ 1 item(s) from $\mathcal{L}$

Example:



Very Brief Analysis – Main Ideas

Observation: for $\mathcal{DD} = \mathcal{D}$, every defective item i must not be masked by $\mathcal{PD} \setminus \{i\}$.

Idea: split $\mathcal{PD} \setminus \{i\}$ into two sets $\mathcal{D} \setminus \{i\}$ (defective items) & $\mathcal{PD} \setminus \mathcal{D}$ (non-defective items) and consider two error events:

- **Event 1:** #collisions between $i \in \mathcal{D}$ and $\mathcal{D} \setminus \{i\}$ is "close to γ"

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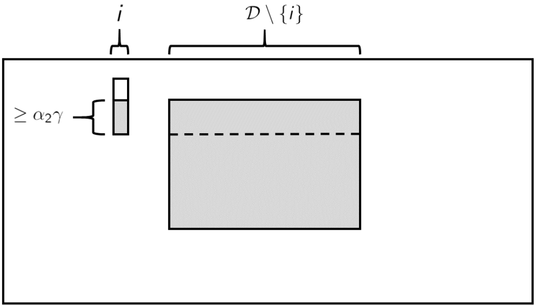

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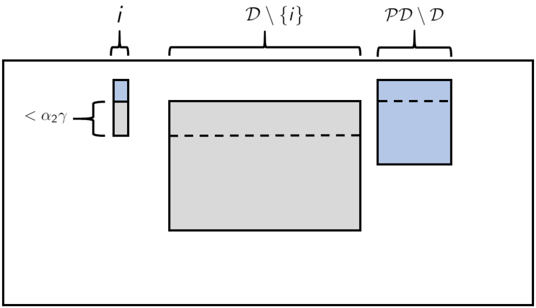
- **Event 2$|\neg$Event1:** every test that includes i as the only defective item, contains ≥ 1 item from $\mathcal{PD} \setminus \mathcal{D}$



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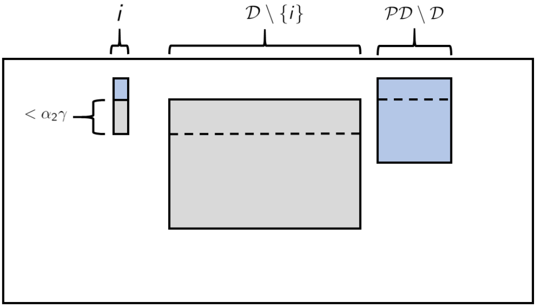

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Interpretation of Results

Result:

$$T = \gamma d \max \left\{ 2^{\frac{1}{\alpha_2} H_2(\max\{\alpha_2, \frac{1}{2}\})} \left(\frac{d}{\beta_n}\right)^{\frac{1}{\alpha_2 \gamma}}, 2^{1/\gamma} \left(\frac{n-d}{d}\right)^{\frac{1}{\gamma}} \left(\frac{d}{\beta_n}\right)^{\frac{1}{(1-\alpha_2)\gamma^2}} \right\}$$

We can simplify the result under two scaling regimes where β_n scales logarithmically in n.

- Large γ: $\gamma \in \Theta\big((\log n)^c\big)$ for some $c \in (0, 1)$
  - Assume that $\alpha_2 \approx 1$
  - $T = \widetilde{\Omega}\big(\gamma d \max\{n^{\theta}, n^{1-\theta}\}^{1/\gamma}\big)$
- Constant γ: $\gamma \in O(1)$
  - $T = \widetilde{\Omega}\Big(\gamma d \max\Big\{n^{\frac{\theta}{\alpha_2 \gamma}}, n^{\frac{1}{\alpha_2 \gamma} + \frac{\theta}{(1-\alpha_2)\gamma^2)}}\Big\}^{1/\gamma}\Big)$
  - We can numerically optimize α_2 to obtain a plot

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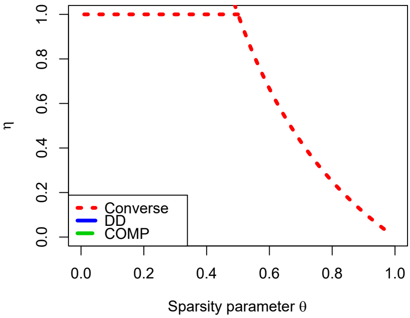
- Constant γ: $\gamma \in O(1)$
  - $T = \widetilde{\Omega}\big(\gamma d \max\big\{n^{\frac{\theta}{\alpha_2 \gamma}}, n^{\frac{1}{\alpha_2 \gamma} + \frac{\theta}{(1-\alpha_2)\gamma^2)}}\big\}^{1/\gamma}\big)$
  - We can numerically optimize α_2 to obtain a plot

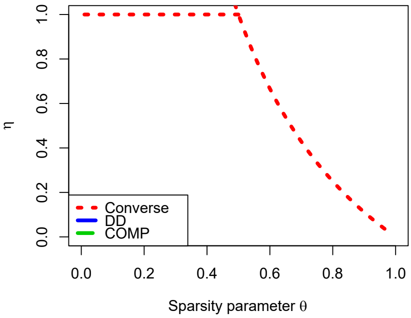
Interpretation of Results

- **Large** γ**:** $T = \widetilde{\Omega}\big(\gamma d \max\{n^{\theta}, n^{1-\theta}\}^{1/\gamma}\big)$ vs. $T = \widetilde{\Omega}\big(\gamma d n^{1/\gamma}\big)$
- **Constant** γ**:** $T = \widetilde{\Omega}\Big(\gamma d \max\Big\{n^{\frac{\theta}{\alpha_2\gamma}}, n^{\frac{1}{\alpha_2\gamma}+\frac{\theta}{(1-\alpha_2)\gamma^2)}}\Big\}^{1/\gamma}\Big)$ vs. $T = \widetilde{\Omega}\big(\gamma d n^{1/\gamma}\big)$

Large γ: $\gamma \in \Theta\big((\log n)^c\big)$



Constant γ: $\gamma \in O(1)$



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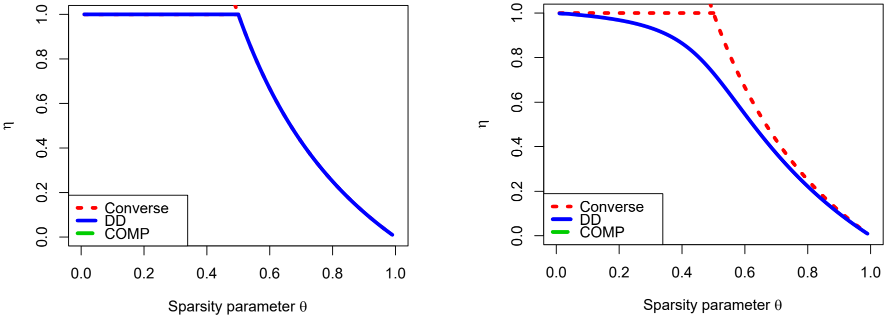
Large γ: $\gamma \in \Theta\big((\log n)^c\big)$ Constant γ: $\gamma \in O(1)$



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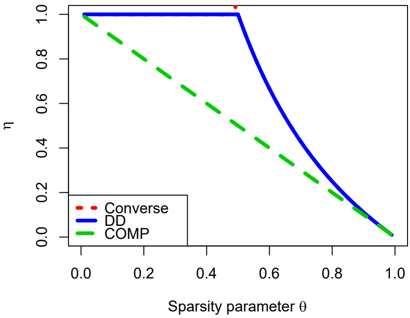
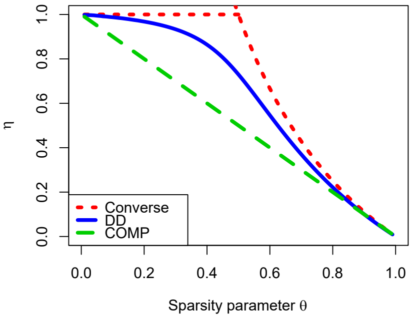


Constant γ: $\gamma \in O(1)$



3 Conclusion

Recap

We focused on:

- γ-divisible items constraints
- Both the adaptive and non-adaptive settings

Returning to the **central problem**:

- How many tests are required? Provided lower bounds
- How can it be achieved? Provided algorithms and their upper bounds

Recap

We focused on:

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Overview of Results

	Scaling regime	Tests required
adaptive	$d \in o(n)$ $\gamma \in o(\log n)$ $\gamma d \to \infty$	$T > e^{-(1+o(1))}\gamma d\left(\frac{n}{d}\right)^{1/\gamma}$ $T < \gamma d\left(\frac{n}{d}\right)^{1/\gamma}$
non-adaptive	$d \in \Theta(n^{\theta}), \theta < 1$ $\gamma \in \Theta\big((\log n)^{c}\big), c < 1$	$T > \gamma d^{\frac{1}{\gamma}}(d-1)(1+o(1))$ $T < \gamma d \max\left\{2^{\frac{1}{\alpha_2}H_2(\max\{\alpha_2,\frac{1}{2}\})}\left(\frac{d}{\beta_n}\right)^{\frac{1}{\alpha_2\gamma}}, 2^{1/\gamma}\left(\frac{n-d}{d}\right)^{\frac{1}{\gamma}}\left(\frac{d}{\beta_n}\right)^{\frac{1}{(1-\alpha_2)\gamma^2}}\right\}$

Table: Our results

Overview of Results

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adaptive	$d \in o(n)$ $\gamma \in o(\log n)$ $\gamma d \to \infty$	$T > e^{-(1+o(1))}\gamma d\left(\frac{n}{d}\right)^{1/\gamma}$ $T < \gamma d\left(\frac{n}{d}\right)^{1/\gamma}$
non-adaptive	$d \in \Theta(n^{\theta}), \theta < 1$ $\gamma \in \Theta\big((\log n)^{c}\big), c < 1$	$T > \gamma d^{\frac{1}{\gamma}}(d-1)(1+o(1))$ $T < \gamma d \max\left\{2^{\frac{1}{\alpha_2}H_2(\max\{\alpha_2, \frac{1}{2}\})}\left(\frac{d}{\beta_n}\right)^{\frac{1}{\alpha_2\gamma}},\right.$ $\left. 2^{1/\gamma}\left(\frac{n-d}{d}\right)^{\frac{1}{\gamma}}\left(\frac{d}{\beta_n}\right)^{\frac{1}{(1-\alpha_2)\gamma^2}}\right\}$

Table: Our results